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The Gauss–Bonnet–Chern mass of higher codimension graphical manifolds

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Abstract

We give an explicit formula for the Gauss–Bonnet–Chern mass of an asymptotically flat graphical manifold of arbitrary codimension and use it to prove the positive mass theorem and the Penrose inequality for graphs with flat normal bundle.

1 Introduction

A complete Riemannian manifold (M^n, g), $n \geq 3$, is said to be asymptotically flat of order τ (with one end) if there exists a compact subset K of M and a diffeomorphism $\Psi : M \setminus K \to \mathbb{R}^n \setminus \overline{B}_1(0)$, introducing coordinates in $M \setminus K$, say $x = (x_1, \ldots, x_n)$, such that, in these coordinates,

$$g_{ij} = \delta_{ij} + \sigma_{ij} \tag{1}$$

and

$$|\sigma_{ij}| + |x||\sigma_{ij,k}| + |x|^2|\sigma_{ij,kl}| = O(|x|^{-\tau}), \tag{2}$$

where the σ_{ij}'s are the coefficients of σ with respect to x, $\sigma_{ij,k} = \partial\sigma_{ij}/\partial x_k$, $\sigma_{ij,kl} = \partial^2\sigma_{ij}/\partial x_k \partial x_l$, and $|\ |$ is the standard Euclidean norm. The ADM mass of (M, g), introduced by Arnowitt, Deser and Misner in [3] (see also [19]) is defined by

$$\mathrm{m}_{\mathrm{ADM}} = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu^j dS_r, \tag{3}$$

where ω_{n-1} is the volume of the unit sphere of dimension $(n-1)$, S_r is the Euclidean coordinate sphere of radius r, dS_r is the volume form of S_r induced by the Euclidean metric, and $\nu = r^{-1}x$ is the outward unit normal to S_r.

It is known that if $\tau > (n-2)/2$ and the scalar curvature of (M, g) is integrable, then the limit (3) exists, is finite, and is a geometric invariant, that is, two coordinate systems satisfying (1) and (2) yield the same value for it [4, 9].

One of the most important conjectures in Mathematical General Relativity is the famous Positive Mass Conjecture (PMC):

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Conjecture 1. *If (M^n, g), $n \geq 3$, is an asymptotically flat Riemannian manifold of order $\tau > (n-2)/2$ whose scalar curvature is nonnegative and integrable, than the ADM mass of (M, g) is nonnegative. Moreover, if the mass is zero, than (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, \delta)$.*

The PMC was settled by Schoen and Yau when $n \leq 7$ [28] and when (M, g) is conformally flat [29], and by Witten when M is spin [30] (see also [25] and [8]). Very elegant proofs for the case when (M, g) is an Euclidean graph were given by Lam [22] (see also [10]) for graphs of codimension one and by Mirandola and Vitório [24] for graphs of arbitrary codimension with flat normal bundle (notice that the case of graphs also follows from Witten's argument, since an Euclidean graph is spin). The rigidity statement for Euclidean hypersurfaces was proved in [16], under appropriate conditions (see [16], Section 5).

The Penrose Inequality (PI) is a conjectured sharpening of the PMC when (M, g) has a compact boundary Γ which is an outermost minimal hypersurface.

Conjecture 2. *If (M^n, g), $n \geq 3$, is an asymptotically flat Riemannian manifold of order $\tau > (n-2)/2$ whose scalar curvature is nonnegative (and integrable), and Γ is a (possibly disconnected) outermost minimal hypersurface of area A, then*

$$\mathrm{m}_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}.$$

Moreover, if the equality holds, then (M, g) is isometric to the Riemannian Schwarzschild manifold.

The PI was proved by Huisken and Ilmanen [18] for $n = 3$ and Γ connected, and by Bray [5] for $n = 3$ and general Γ. In [6], Bray and Lee established the conjecture when $n \leq 7$, with the extra requirement that M be spin for the rigidity statement. The case of Euclidean graphs of codimension one was treated by Lam in [22] (see also [10]) and generalized by Mirandola and Vitório for graphs of arbitrary codimension with flat normal bundle [24]. The equality case for graphs of codimension one was treated in [17].

In [13], a new mass for asymptotically flat Riemannian manifolds, named Gauss–Bonnet–Chern mass, was introduced. For a positive integer $q \leq n/2$, consider the q-th Gauss–Bonnet curvature, denoted $L_{(q)}$, and defined by

$$L_{(q)} = \frac{1}{2^q}\delta^{a_1 a_2 \cdots a_{2q}}_{b_1 b_2 \cdots b_{2q}}\left(\prod_{s=1}^{q} R_{a_{2s-1}a_{2s}}{}^{b_{2s-1}b_{2s}}\right) = P_{(q)}^{ijkl} R_{ijkl}, \tag{4}$$

where R is the Riemann curvature tensor of (M, g) and $P_{(q)}$, which has the same symmetries of the Riemann tensor (see [13], Section 3), is given by

$$P_{(q)}^{ijkl} = \frac{1}{2^q}\delta^{a_1 a_2 \cdots a_{2q-3} a_{2q-2} ij}_{b_1 b_2 \cdots b_{2q-3} b_{2q-2} b_{2q-1} b_{2q}}\left(\prod_{s=1}^{q-1} R_{a_{2s-1}a_{2s}}{}^{b_{2s-1}b_{2s}}\right) g^{b_{2q-1}k} g^{b_{2q}l}. \tag{5}$$

Remark 1. One can considerably simplify this complicated tensorial expression by rewriting it in the language of double forms, which are a special type of vector valued forms (see [21], for example).

The q-th Gauss–Bonnet–Chern mass (GBC mass) of (M, g) is defined by

$$\mathrm{m}_q = c_q(n) \lim_{r\to\infty} \int_{S_r} P_{(q)}^{ijkl} g_{jk,l} \nu_i dS_r, \tag{6}$$

where

$$c_q(n) = \frac{(n-2q)!}{2^{q-1}(n-1)!\omega_{n-1}} \tag{7}$$

and S_r, dS_r, ν, ω_{n-1} are as in the definition of the ADM mass.

As observed in [13], m_1 coincides with the ADM mass. In the same article, the authors show that, if $\tau > (n-2q)/(q+1)$ and $L_{(q)}$ is integrable, then the limit (6) exists, is finite, and is a geometric invariant. Next, we state versions of the PMC and PI for the GBC mass. We start with the version of the PMC.

Conjecture 3. *Let n and q be integers such that $n \geq 3$ and $1 \leq q < n/2$. If (M^n, g) is an asymptotically flat Riemannian manifold of order $\tau > (n-2q)/(q+1)$ whose q-th Gauss–Bonnet curvature $L_{(q)}$ is nonnegative and integrable, than the q-th GBC mass of (M, g) is nonnegative. Moreover, if the mass is zero, than (M, g) is isometric to the Euclidean space $(\mathbb{R}^n, \delta)$.*

Before we state the analogue of the PI, we recall the Riemannian manifold known as the q-th Riemannian Schwarzschild [13, Section 6], which is $(\mathbb{R} \times \mathbb{S}^{n-1}, g^q_{\mathrm{Sch}})$ with

$$g^q_{\mathrm{Sch}} = \left(1 + \frac{m}{2r^{\frac{n}{q}-2}}\right)^{\frac{4q}{n-2q}} \left(dr^2 + r^2 d\theta^2\right), \tag{8}$$

where $d\theta^2$ is the round metric on $\mathbb{S}^{n-1}$ and $m \in \mathbb{R}$ is the mass parameter. Let $r_0 = (2m)^{\frac{q}{n-2q}}$. The hypersurface $r = r_0$ is an outermost minimal hypersurface of area $A = \omega_{n-1} r_0^{n-1}$, and the q-th GBC mass of $(\mathbb{R} \times \mathbb{S}^{n-1}, g^q_{\mathrm{Sch}})$ is $\mathrm{m}_q = m^q$. Thus, for the q-th Riemannian Schwarzschild manifold, one has

$$\mathrm{m}_q = \frac{1}{2^q} \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2q}{n-1}}.$$

We can now state the version of the PI for the GBC mass.

Conjecture 4. *Let n and q be integers such that $n \geq 3$ and $1 \leq q < n/2$. If (M^n, g) is an asymptotically flat Riemannian manifold of order $\tau > (n-2q)/(q+1)$ whose q-th Gauss–Bonnet curvature $L_{(q)}$ is nonnegative and integrable, and Γ is a (possibly disconnected) outermost minimal hypersurface of area A, then*

$$\mathrm{m}_q \geq \frac{1}{2^q} \left(\frac{A}{\omega_{n-1}}\right)^{\frac{n-2q}{n-1}},$$

where m_q is the q-th GBC mass. Moreover, if the equality holds, then (M, g) is isometric to the q-th Riemannian Schwarzschild manifold.

We now turn to the special case of graphs. Let Ω be a (possibly empty) bounded open subset of $\mathbb{R}^n$ such that $\Sigma = \partial\Omega$ is the union of finitely many smooth hypersurfaces. Let $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$ be a continuous map such that its restriction to $\mathbb{R}^n \setminus \overline{\Omega}$ is smooth. Let f^α, $1 \leq \alpha \leq m$, be the components of f and let f_i^α, f_{ij}^α and f_{ijk}^α denote the first, second and third partial derivatives of f^α on $\mathbb{R}^n \setminus \overline{\Omega}$, where $1 \leq i, j, k \leq n$. The map f is said to be asymptotically flat of order τ if

$$|f_i^\alpha(x)| + |f_{ij}^\alpha(x)||x| + |f_{ijk}^\alpha(x)||x|^2 = O(|x|^{-\tau/2}), \tag{9}$$

for each $\alpha \in \{1, \ldots, m\}$ and each triple (i, j, k) with $1 \leq i, j, k \leq n$.

We assume throughout the paper that

$$M = \{(x, f(x));\ x \in \mathbb{R}^n \setminus \Omega, f(x) \in \mathbb{R}^m\},$$

the graph of f, is a smooth submanifold with (possibly empty) boundary and that $g_{ij} = \delta_{ij} + f_i^\alpha f_j^\alpha$, the metric induced by the Euclidean metric on $\mathbb{R}^{n+m}$, extends to a smooth metric on M. Notice that if f is asymptotically flat of order τ, than from (9) we get that (M, g) is asymptotically flat of order τ.

Conjectures 3 and 4 were proved for graphs of codimension one [13]. When $q = 2$, Li, Wei and Xiong proved these conjectures for graphs of higher codimension with flat normal bundle [23]. Conjecture 3 is also known to be true for conformally flat manifolds [12].

The purpose of the present article is to prove conjectures 3 and 4 for a family of higher codimension Euclidean graphs (without the rigidity statements). This family includes the graphs with flat normal bundle. The exposition follows closely the ones given in [22], [24], [13] and [23]. Before stating our main results, we need to introduce some notation.

Denote by $\{e_i\}_{i=1}^n$ the standard basis of $\mathbb{R}^n$ and by $\{e_\alpha\}_{\alpha=1}^m$ the standard basis of $\mathbb{R}^m$. The coordinate vector fields on M are given by $\partial_i = (e_i, f_i^\alpha e_\alpha)$, and the vector fields $\eta_\alpha = (-Df^\alpha, e_\alpha)$, where Df^α denotes the Euclidean gradient of f^α, give us a (global) frame field for the normal bundle of M. We denote by B the second fundamental form of M, by B_α its α-th component with respect to the frame $\{\eta_\alpha\}_{\alpha=1}^n$, and by A_α the shape operator with respect to η_α. Also, let $U = (U_{\alpha\beta})$ be the metric on the normal bundle induced by the Euclidean metric $\langle\ ,\ \rangle$ on $\mathbb{R}^{n+m}$. The components of U are given by

$$U_{\alpha\beta} = \delta_{\alpha\beta} + \langle Df^\alpha, Df^\beta \rangle.$$

The inverse of U is denoted by $(U^{\alpha\beta})$.

Recall the Gauss and the Ricci equations, which are respectively given by

$$R_{ijkl} = \langle B_{ik}, B_{jl} \rangle - \langle B_{il}, B_{jk} \rangle \tag{10}$$

and

$$\langle R^\perp_{\alpha\beta}(X), Y \rangle = \langle [A_\beta, A_\alpha](X), Y \rangle, \tag{11}$$

where $\langle\ ,\ \rangle$ is the Euclidean metric on $\mathbb{R}^{n+m}$, $R^\perp$ is the normal curvature operator and $[A_\beta, A_\alpha] = A_\alpha \circ A_\beta - A_\beta \circ A_\alpha$.

We denote by $T_{(2q-1)}$ the Newton tensor of order $(2q-1)$ and denote by $T_{(2q-1)\alpha}$ its α-th component with respect to the frame $\{\eta_\alpha\}_{\alpha=1}^n$ (see [27], [14] and [7]). The expression for $T_{(2q-1)}$ in coordinates is

$$T_{(2q-1)i}{}^{j} = \frac{1}{(2q-1)!}\delta^{a_1\cdots a_{2q-1}j}_{b_1\cdots b_{2q-1}i}\langle B^{b_1}_{a_1}, B^{b_2}_{a_2}\rangle \cdots \langle B^{b_{2q-3}}_{a_{2q-3}}, B^{b_{2q-2}}_{a_{2q-2}}\rangle B^{b_{2q-1}}_{a_{2q-1}}, \tag{12}$$

where $\langle\ ,\ \rangle$ denotes the Euclidean metric on $\mathbb{R}^{n+m}$. As we will see in Section 2, if M has flat normal bundle, then $T_{(2q-1)\alpha}$ commutes with A_β, for $1 \leq \alpha, \beta \leq m$.

We can now state the main results of the article. Our first main result is the following:

Theorem 1. *Let n and q be integers such that $n \geq 3$ and $1 \leq q < n/2$, and let (M, g) be the graph of an asymptotically flat map $f : \mathbb{R}^n \to \mathbb{R}^m$ of order $\tau > (n-2q)/(q+1)$. If the q-th Gauss–Bonnet curvature $L_{(q)}$ of (M, g) is integrable, then the q-th Gauss–Bonnet–Chern mass m_q satisfies*

$$\mathrm{m}_q = \frac{1}{2}c_q(n)\int_M \left(L_{(q)} + (2q-1)!\left\langle \left[T_{(2q-1)\alpha}, A_\beta\right]\cdot e_\alpha^\top, e_\beta^\top\right\rangle\right)\frac{1}{\sqrt{G}}dM, \tag{13}$$

where $c_q(n)$ is the constant (7), G is the determinant of (g_{ij}), $\left[T_{(2q-1)\alpha}, A_\beta\right] = T_{(2q-1)\alpha}\circ A_\beta - A_\beta \circ T_{(2q-1)\alpha}$ is the commutator of the operators $T_{(2q-1)\alpha}$ and A_β, and $e_\alpha^\top$ is the tangent part (along of graph M) of the canonical lift to $\mathbb{R}^{n+m} \equiv \mathbb{R}^n \times \mathbb{R}^m$ of the standard frame field on $\mathbb{R}^m$. Moreover, if M has flat normal bundle and $L_{(q)}$ is nonnegative, then m_q is nonnegative.

Let $\Sigma \subset \mathbb{R}^n$ be an orientable hypersurface and let ξ be a unit normal vector field along Σ (chosen to point outwards, whenever this makes sense). The r-th mean curvature of Σ is defined as the r-th elementary symmetric function on the principal curvatures of Σ. Alternatively, if K is the second fundamental form of $\Sigma \subset \mathbb{R}^n$, then

$$H_r = \frac{1}{r!}\delta^{a_1\cdots a_r}_{b_1\cdots b_r}\prod_{s=1}^{r} K^{b_s}_{a_s}. \tag{14}$$

The hypersurface $\Sigma \subset \mathbb{R}^n$ is called *strictly p-mean convex*, $1 \leq p \leq n-1$, if $H_r > 0$ for all $1 \leq r \leq p$. Our second main result is the following:

Theorem 2. *Let n and q be integers such that $n \geq 3$ and $1 \leq q < n/2$. Let Ω be a bounded and open subset of $\mathbb{R}^n$ such that $\Sigma = \partial\Omega$ is the union of finitely many smooth hypersurfaces. Let $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$ be an asymptotically flat map of order $\tau > (n-2q)/(q+1)$, and let (M, g) be the graph of f. Assume that f extends smoothly to an open set containing $\mathbb{R}^n \setminus \Omega$ and that f is constant along each connected component of Σ. If the q-th Gauss–Bonnet curvature $L_{(q)}$ is integrable, then the q-th Gauss–Bonnet–Chern mass m_q satisfies*

$$\begin{aligned}\mathrm{m}_q =& \frac{1}{2}c_q(n)\int_M \left(L_{(q)} + (2q-1)!\left\langle \left[T_{(2q-1)\alpha}, A_\beta\right]\cdot e_\alpha^\top, e_\beta^\top\right\rangle\right)\frac{1}{\sqrt{G}}dM \\ &+ \frac{1}{2}(2q-1)!c_q(n)\int_\Sigma \left(\frac{|Df|^2}{1+|Df|^2}\right)^q H_{(2q-1)}d\Sigma,\end{aligned} \tag{15}$$

where

$$|Df|^2 = \sum_{\alpha=1}^{m} |Df^\alpha|^2$$

and $H_{(2q-1)}$ *is the* $(2q-1)$*-th mean curvature of* Σ*.*

Our third main result is the following:

Theorem 3. *Let* n *and* q *be integers such that* $n \geq 3$ *and* $1 \leq q < n/2$*. Let* Ω *be a bounded and open subset of* $\mathbb{R}^n$ *such that* $\Sigma = \partial\Omega$ *is the union of finitely many smooth hypersurfaces. Let* $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m$ *be an asymptotically flat map of order* $\tau > (n-2q)/(q+1)$*, and let* (M, g) *be the graph of* f*. Assume that* f *is constant along each connected component of* Σ *and that*

$$|Df| \to \infty \quad as \quad x \to \Sigma.$$

If the q-th Gauss-Bonnet curvature $L_{(q)}$ *is integrable, then the q-th Gauss-Bonnet-Chern mass* m_q *satisfies*

$$\begin{aligned} \mathrm{m}_q =& \frac{1}{2} c_q(n) \int_M \left(L_{(q)} + (2q-1)! \left\langle \left[T_{(2q-1)\alpha}, A_\beta \right] \cdot e_\alpha^\top, e_\beta^\top \right\rangle \right) \frac{1}{\sqrt{G}} dM \\ &+ \frac{1}{2}(2q-1)! c_q(n) \int_\Sigma H_{(2q-1)} d\Sigma, \end{aligned} \tag{16}$$

where $H_{(2q-1)}$ *is the* $(2q-1)$*-th mean curvature of* Σ*. Furthermore, if* M *has flat normal bundle,* $L_{(q)}$ *is nonnegative and each component of* Σ *is star-shaped and strictly* $(2q-1)$*-mean convex, then*

$$\mathrm{m}_q \geq \frac{1}{2^q} \left(\frac{A}{\omega_{n-1}} \right)^{\frac{n-2q}{n-1}}. \tag{17}$$

Remark 2. As explained in [13] (Remark 5.1), when $\Sigma \subset \mathbb{R}^n$ is stricly mean convex, the condition

$$|Df| \to \infty \quad \text{as} \quad x \to \partial\Omega \tag{18}$$

holds if and only if $\Gamma = \partial M$ is an outermost minimal hypersurface. Therefore, this is a natural assumption.

Remark 3. Geometrically, condition (18) is equivalent to saying that along each connected component of ∂M, the graph M meets orthogonally the hyperplane that contains that component (see [11]).

2 Auxiliary results

Let n and q be positive integers such that $n \geq 3$ and $1 \leq q < n/2$. Throughout this section, the tensors $P_{(q)}$ and $T_{(2q-1)}$ are defined by equations (5) and (12), respectively. Also, unless stated otherwise, we will follow the notation introduced in Section 1.

If Ω is not empty, we assume, throughout this section, that f extends smoothly to an open set containing $\mathbb{R}^n \setminus \Omega$.

Lemma 1. *Under the notation introduced above, the following identities hold:*

$$g_{jk,l} = f^\alpha_{jl} f^\alpha_k + f^\alpha_j f^\alpha_{kl} \tag{19}$$

$$e^\top_\alpha = \nabla f^\alpha = g^{ij} f^\alpha_j \partial_i = U^{\alpha\beta} f^\beta_i \partial_i \tag{20}$$

$$A_\alpha \partial_i = f^\alpha_{ik} g^{kj} \partial_i \tag{21}$$

$$B(\partial_i, \partial_j) = f^\alpha_{ij} U^{\alpha\beta} \eta^\beta \tag{22}$$

$$(B_\alpha)_{ij} = f^\alpha_{ij} \tag{23}$$

$$\Gamma^k_{ij} = g^{kl} f^\alpha_l f^\alpha_{ij} \tag{24}$$

Proof. These identities are proven in [24] and [23]. □

On an open set that contains $\mathbb{R}^n \setminus \Omega$, consider the vector field $X_{(q)}$ given by

$$X_{(q)} = X^i_{(q)} \partial_i = P^{ijkl}_{(q)} g_{jk,l} \partial_i. \tag{25}$$

Proposition 1. *It holds*

$$X_{(q)} = \frac{1}{2}(2q-1)!\ T_{(2q-1)\alpha} \cdot e^\top_\alpha. \tag{26}$$

Proof. By (25) and (19) we have

$$X^i_{(q)} = P^{ijkl}_{(q)} \left(f^\alpha_{jl} f^\alpha_k + f^\alpha_j f^\alpha_{kl} \right).$$

Using the antisymmetry of $P^{ijkl}_{(q)}$ with respect to the indices k and l, we have

$$X^i_{(q)} = P^{ijkl}_{(q)} f^\alpha_{jl} f^\alpha_k.$$

Combining this identity with (5), (21) and (20), we find

$$\begin{aligned}
X^i_{(q)} &= \frac{1}{2^q} \delta^{a_1 a_2 \cdots a_{2q-3} a_{2q-2} ij}_{b_1 b_2 \cdots b_{2q-3} b_{2q-2} cd} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}{}^{b_{2s-1} b_{2s}} \right) g^{ck} g^{dl} f^\alpha_{jl} f^\alpha_k \\
&= \frac{1}{2^q} \delta^{a_1 a_2 \cdots a_{2q-3} a_{2q-2} ij}_{b_1 b_2 \cdots b_{2q-3} b_{2q-2} cd} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}{}^{b_{2s-1} b_{2s}} \right) g^{dl} f^\alpha_{jl} g^{ck} f^\alpha_k \\
&= \frac{1}{2^q} \delta^{a_1 a_2 \cdots a_{2q-3} a_{2q-2} ij}_{b_1 b_2 \cdots b_{2q-3} b_{2q-2} cd} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}{}^{b_{2s-1} b_{2s}} \right) (A_\alpha)^d_j\, g^{ck} f^\alpha_k \\
&= \frac{1}{2^q} \delta^{a_1 a_2 \cdots a_{2q-3} a_{2q-2} ij}_{b_1 b_2 \cdots b_{2q-3} b_{2q-2} cd} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}{}^{b_{2s-1} b_{2s}} \right) (A_\alpha)^d_j\, (\nabla f^\alpha)^c.
\end{aligned}$$

Hence, using (10), (12), (20) and switching i with j and c with d, we find

$$\begin{aligned}
X^i_{(q)} &= \frac{2^{q-1}}{2^q}\delta^{a_1a_2\cdots a_{2q-3}a_{2q-2}ij}_{b_1b_2\cdots b_{2q-3}b_{2q-2}cd}\left(\prod_{s=1}^{q-1}\langle B^{b_{2s-1}}_{a_{2s-1}}, B^{b_{2s}}_{a_{2s}}\rangle\right)(A_\alpha)^d_j\,(\nabla f^\alpha)^c \\
&= \frac{1}{2}\delta^{a_1a_2\cdots a_{2q-3}a_{2q-2}ji}_{b_1b_2\cdots b_{2q-3}b_{2q-2}dc}\left(\prod_{s=1}^{q-1}\langle B^{b_{2s-1}}_{a_{2s-1}}, B^{b_{2s}}_{a_{2s}}\rangle\right)(A_\alpha)^d_j\,(\nabla f^\alpha)^c \\
&= \frac{1}{2}(2q-1)!\left(T_{(2q-1)\alpha}\right)^i_c(\nabla f^\alpha)^c \\
&= \frac{1}{2}(2q-1)!\left(T_{(2q-1)\alpha}\cdot\nabla f^\alpha\right)^i \\
&= \frac{1}{2}(2q-1)!\left(T_{(2q-1)\alpha}\cdot e^\top_\alpha\right)^i.
\end{aligned}$$

□

The next identity is a higher codimensional version of Proposition 3.5 (b) in [26] (see also [1], Section 8).

Proposition 2. *It holds*

$$div_e X = \frac{1}{2}L_{(q)} + \frac{1}{2}(2q-1)!\left\langle\left[T_{(2q-1)\alpha}, A_\beta\right]\cdot e^\top_\alpha, e^\top_\beta\right\rangle, \tag{27}$$

where div_e denotes the Euclidean divergence.

Proof. Using the identity

$$div_e X_{(q)} = \partial_i X^i_{(q)}$$

and the identities (20), (23) and (24), we have

$$\begin{aligned}
div_g X_{(q)} &= \nabla_i X^i_{(q)} = \partial_i X^i_{(q)} + \Gamma^i_{ij}X^j_{(q)} \\
&= div_e X_{(q)} + (e^\top_\beta)^i (B_\beta)_{ij} X^j_{(q)} \\
&= div_e X_{(q)} + \left\langle A_\beta\cdot X_{(q)}, e^\top_\beta\right\rangle \\
&= div_e X_{(q)} + \frac{1}{2}(2q-1)!\left\langle\left(A_\beta\circ T_{(2q-1)\alpha}\right)\cdot e^\top_\alpha, e^\top_\beta\right\rangle.
\end{aligned}$$

By the expression for the vector field $X_{(q)}$ established in the previous proposition, it follows that

$$\begin{aligned}
div_g X_{(q)} &= \frac{1}{2}(2q-1)!\; div_g\left(T_{(2q-1)\beta}\cdot e^\top_\beta\right) \\
&= \frac{1}{2}(2q-1)!\left[div_g\left(T_{(2q-1)\beta}\right)\cdot e^\top_\beta + T_{(2q-1)\beta}\cdot\nabla e^\top_\beta\right].
\end{aligned}$$

Using identity (20) and the fact that $f^\beta = \langle f, e_\beta\rangle$ is the height function with respect to the Euclidean hyperplane normal to the vector e_β, the identities

$$\nabla e^\top_\beta = \nabla^2 f^\beta = \langle B, e_\beta\rangle = U^{\gamma\alpha}\langle\eta_\gamma, e_\beta\rangle B_\alpha = U^{\beta\alpha}B_\alpha$$

hold. Therefore, the Gauss equation together with identities (12) and (4) give

$$T_{(2q-1)\beta} \cdot \nabla e_\beta^\top = U^{\beta\alpha} T_{(2q-1)\beta} \cdot B_\alpha = \frac{1}{(2q-1)!} L_{(q)}. \tag{28}$$

Thus,

$$div_e X_{(q)} = \frac{1}{2} L_{(q)} + \frac{1}{2}(2q-1)! \left[div_g \left(T_{(2q-1)\beta} \right) \cdot e_\beta^\top - \left\langle \left(A_\beta \circ T_{(2q-1)\alpha} \right) \cdot e_\alpha^\top, e_\beta^\top \right\rangle \right].$$

Recall that the Newton tensors of a submanifold of Euclidean space are divergence free (see, for example, lemmata 3.1 and 3.2 of [7]) and that each of the fields in the normal frame is given by the expression $\eta_\beta = (-Df^\beta, e_\beta)$. Hence, using identity (20), we have

$$\begin{aligned}
\left(div_g T_{(2q-1)\beta} \right)_j &= \nabla_i \left(T_{(2q-1)\beta} \right)^i{}_j = \nabla_i \left\langle \left(T_{(2q-1)} \right)^i{}_j, \eta_\beta \right\rangle \\
&= \left\langle \nabla_i^\perp \left(T_{(2q-1)} \right)^i{}_j, \eta_\beta \right\rangle + \left\langle \left(T_{(2q-1)} \right)^i{}_j, \nabla_i^\perp \eta_\beta \right\rangle \\
&= \left\langle \left(\mathrm{div}\ T_{(2q-1)} \right)_j, \eta_\beta \right\rangle + U^{\gamma\alpha} \left(T_{(2q-1)\alpha} \right)^i{}_j \langle \eta_\gamma, \bar{D}_i \eta_\beta \rangle \\
&= \left(T_{(2q-1)\alpha} \right)^i{}_j U^{\gamma\alpha} f_k^\gamma f_{ik}^\beta = \left(T_{(2q-1)\alpha} \right)^i{}_j \left(e_\alpha^\top \right)^k (B_\beta)_{ik} \\
&= \left(\left(T_{(2q-1)\alpha} \circ A_\beta \right) \cdot e_\alpha^\top \right)_j .
\end{aligned}$$

Therefore,

$$\begin{aligned}
div_e X_{(q)} &= \frac{1}{2} L_{(q)} + \frac{1}{2}(2q-1)! \left\langle \left(T_{(2q-1)\alpha} \circ A_\beta - A_\beta \circ T_{(2q-1)\alpha} \right) \cdot e_\alpha^\top, e_\beta^\top \right\rangle \\
&= \frac{1}{2} L_{(q)} + \frac{1}{2}(2q-1)! \left\langle \left[T_{(2q-1)\alpha}, A_\beta \right] \cdot e_\alpha^\top, e_\beta^\top \right\rangle .
\end{aligned}$$

□

Proposition 3. *For a level set $\Sigma \subset \mathbb{R}^n$ in the domain of a euclidean graph, the identity*

$$\langle X_{(q)}, \xi \rangle = -\frac{1}{2}(2q-1)! \left(\frac{|Df|^2}{1+|Df|^2} \right)^q H_{(2q-1)}$$

holds, where ξ denotes a unit normal vector field along Σ (chosen to point outwards, whenever this makes sense).

Proof. We have

$$\langle X_{(q)}, \xi \rangle = \frac{1}{2}(2q-1)! \left(T_{(2q-1)\alpha} \cdot e_\alpha^\top \right)^i \xi_i.$$

Let $x \in \Sigma$. Rotate the coordinates such that, at x, $e_1 = \xi$ and $\{e_A\}_{A=2}^n$ is an orthonormal frame for the tangent space of Σ at x. With respect to this new frame $\{e_i\}_{i=1}^n$ on $\mathbb{R}^n$,

$$\xi_i = \delta_i^1,$$

for $i = 1, \ldots, n$. Thus,

$$\langle X_{(q)}, \xi \rangle = \frac{1}{2}(2q-1)! \left(T_{(2q-1)\alpha} \cdot e_\alpha^\top \right)^1 .$$

As in section 4 of [23], we find that the inverse of g is given by

$$g^{11} = \frac{1}{1+|Df|^2}, \quad g^{A1} = 0, \text{ and } g^{AB} = \delta^{AB}.$$

It follows that

$$e_\alpha^\top = \frac{f_1^\alpha}{1+|Df|^2}\partial_1 = \frac{\langle Df^\alpha, \xi\rangle}{1+|Df|^2}\partial_1.$$

Therefore,

$$\langle X_{(q)}, \xi \rangle = \frac{1}{2}(2q-1)! \frac{\langle Df^\alpha, \xi\rangle}{1+|Df|^2} \left(T_{(2q-1)\alpha} \right)_1^1 .$$

Since

$$\left(T_{(2q-1)\alpha} \right)_1^1 = \frac{1}{2^{q-1}} \frac{1}{(2q-1)!} \delta^{a_1 \cdots a_{2q-1} 1}_{b_1 \cdots b_{2q-1} 1} \left(\prod_{s=1}^{q-1} R_{a_{2s-1} a_{2s}}{}^{b_{2s-1} b_{2s}} \right) (A_\alpha)^{b_{2q-1}}_{a_{2q-1}} ,$$

using the antisymmetry of $\delta^{a_1 \cdots a_{2q-1} 1}_{b_1 \cdots b_{2q-1} 1}$ we find that

$$\left(T_{(2q-1)\alpha} \right)_1^1 = \frac{1}{2^{q-1}} \frac{1}{(2q-1)!} \delta^{A_1 \cdots A_{2q-1} 1}_{B_1 \cdots B_{2q-1} 1} \left(\prod_{s=1}^{q-1} R_{A_{2s-1} A_{2s}}{}^{B_{2s-1} B_{2s}} \right) (A_\alpha)^{B_{2q-1}}_{A_{2q-1}} .$$

Recall that the generalized Kronecker delta is a determinant. Using the q-th column to expand it, we find

$$\delta^{A_1 \cdots A_{2q-1} 1}_{B_1 \cdots B_{2q-1} 1} = \delta^{A_1 \cdots A_{2q-1}}_{B_1 \cdots B_{2q-1}}.$$

Hence,

$$\left(T_{(2q-1)\alpha} \right)_1^1 = \frac{1}{2^{q-1}} \frac{1}{(2q-1)!} \delta^{A_1 \cdots A_{2q-1}}_{B_1 \cdots B_{2q-1}} \left(\prod_{s=1}^{q-1} R_{A_{2s-1} A_{2s}}{}^{B_{2s-1} B_{2s}} \right) (A_\alpha)^{B_{2q-1}}_{A_{2q-1}} .$$

Let $\hat{R}$ denote the Riemann curvature tensor of Σ, and denote by K and $\tilde{K}$, respectively, the second fundamental form of Σ as a hypersurface of $\mathbb{R}^n$ and the second fundamental form of $f(\Sigma)$ as a hypersurface of (M, g). By equations (4.3) and (4.4) of [23], we have

$$\tilde{K} = \frac{K}{\sqrt{1+|Df|^2}}. \tag{29}$$

and

$$R_{AB}{}^{CD} = \frac{|Df|^2}{1+|Df|^2} \hat{R}_{AB}{}^{CD}. \tag{30}$$

Thus, plugging into the expression for $\left(T_{(2q-1)\alpha}\right)_1^1$, we find

$$\begin{aligned}\left(T_{(2q-1)\alpha}\right)_1^1 = &\frac{1}{2^{q-1}}\frac{1}{(2q-1)!}\left(\frac{|Df|^2}{1+|Df|^2}\right)^{q-1} \\ &\times \delta^{A_1\cdots A_{2q-1}}_{B_1\cdots B_{2q-1}}\left(\prod_{s=1}^{q-1}\hat{R}_{A_{2s-1}A_{2s}}{}^{B_{2s-1}B_{2s}}\right)(A_\alpha)^{B_{2q-1}}_{A_{2q-1}}.\end{aligned}$$

From

$$\eta^\alpha = e_\alpha - Df^\alpha = e_\alpha - \langle Df^\alpha, \xi\rangle\xi,$$

it follows that

$$(A_\alpha)^B_A = -\langle Df^\alpha, \xi\rangle K^B_A.$$

Also, the Gauss equation applied to $\Sigma \subset \mathbb{R}^n$ yields

$$\hat{R}_{AB}{}^{CD} = K^C_A K^D_B - K^D_A K^C_B.$$

We then conclude that

$$\begin{aligned}\left(T_{(2q-1)\alpha}\right)_1^1 =& -\frac{1}{2^{q-1}}\frac{1}{(2q-1)!}\langle Df^\alpha, \xi\rangle\left(\frac{|Df|^2}{1+|Df|^2}\right)^{q-1} \\ &\times \delta^{A_1\cdots A_{2q-1}}_{B_1\cdots B_{2q-1}}\left(\prod_{s=1}^{q-1}\hat{R}_{A_{2s-1}A_{2s}}{}^{B_{2s-1}B_{2s}}\right)(K)^{B_{2q-1}}_{A_{2q-1}} \\ =& -\frac{1}{(2q-1)!}\langle Df^\alpha, \xi\rangle\left(\frac{|Df|^2}{1+|Df|^2}\right)^{q-1} \\ &\times \delta^{A_1\cdots A_{2q-1}}_{B_1\cdots B_{2q-1}}\left(\prod_{s=1}^{q-1}K^{B_{2s-1}}_{A_{2s-1}}K^{B_{2s}}_{A_{2s}}\right)K^{B_{2q-1}}_{A_{2q-1}} \\ =& \ -\langle Df^\alpha, \xi\rangle\left(\frac{|Df|^2}{1+|Df|^2}\right)^{q-1} H_{(2q-1)},\end{aligned}$$

where we have used the expression (14) to obtain the last equality. It follows that

$$\begin{aligned}\langle X_{(q)}, \xi\rangle &= \frac{1}{2}(2q-1)!\frac{\langle Df^\alpha, \xi\rangle}{1+|Df|^2}\left(T_{(2q-1)\alpha}\right)_1^1 \\ &= -\frac{1}{2}(2q-1)!\frac{\langle Df^\alpha, \xi\rangle^2}{1+|Df|^2}\left(\frac{|Df|^2}{1+|Df|^2}\right)^{q-1} H_{(2q-1)} \\ &= -\frac{1}{2}(2q-1)!\left(\frac{|Df|^2}{1+|Df|^2}\right)^{q} H_{(2q-1)},\end{aligned}$$

where, to obtain the last equality, we have used that

$$Df^\alpha = \langle Df^\alpha, \xi\rangle\xi$$

implies

$$|Df|^2 = \sum_\alpha \langle Df^\alpha, \xi\rangle^2.$$

$\square$

Remark 4. In Proposition 3, the expression $|Df|^2/\left(1+|Df|^2\right)$ is the cosine of the angle between the graph and the hyperplane containing its boundary (see [11]).

3 Proof of the theorems

Suppose first that M has no boundary. Let S_r be an Euclidean coordinate sphere of radius r. By (6), (25) and the divergence theorem we have

$$\begin{aligned} \mathrm{m}_q &= c_q(n) \lim_{r\to\infty} \int_{S_r} X^i_{(q)} \xi_i dS_r \\ &= c_q(n) \int_{\mathbb{R}^n} div_e X_{(q)} dV, \end{aligned}$$

where dV denotes the Euclidean volume form. Thus, invoking Proposition 2 and using that

$$dV = \frac{1}{\sqrt{G}} dM, \tag{31}$$

we find

$$\mathrm{m}_q = \frac{1}{2} c_q(n) \int_M \left(L_{(q)} + (2q-1)! \langle \left[T_{(2q-1)\alpha}, A_\beta\right] \cdot e_\alpha^\top, e_\beta^\top \rangle\right) \frac{1}{\sqrt{G}} dM,$$

which is exactly the first part of Theorem 1.

To prove the second part of Theorem 1, notice that, from equations 3 and 6 of [2], the tensor $T_{(2q-1)\alpha}$ can be written as a polynomial on the A_α's. Also, if M has flat normal bundle, then the Ricci equation (11) yields

$$[A_\alpha, A_\beta] = 0, \tag{32}$$

for all $\alpha, \beta \in \{1, \ldots, m\}$. Thus, using (32) several times, we find

$$\left[T_{(2q-1)\alpha}, A_\beta\right] = 0,$$

for all $\alpha, \beta \in \{1, \ldots, m\}$. Hence, equation (13) becomes

$$\mathrm{m}_q = \frac{1}{2} c_q(n) \int_M L_{(q)} \frac{1}{\sqrt{G}} dM.$$

Therefore, if $L_{(q)}$ is nonnegative, than m_q is nonnegative. This finishes the proof of Theorem 1.

Suppose now that ∂M is not empty and that f can be extended to a smooth map on some open set containing $\mathbb{R}^n \setminus \Omega$. This assumption allows us to use the results of Section 2. Equations (6), (25) and the divergence theorem yield

$$\begin{aligned} \mathrm{m}_q &= c_q(n) \lim_{r\to\infty} \int_{S_r} X^i_{(q)} \nu_i dS_r \\ &= c_q(n) \int_{\mathbb{R}^n\setminus\Omega} div_e X_{(q)} dV - c_q(n) \int_\Sigma \langle X_{(q)}, \xi \rangle d\Sigma. \end{aligned}$$

Invoking Proposition 2, Proposition 3 and equation (31), we get

$$\begin{aligned} \mathrm{m}_q = {}& \frac{1}{2} c_q(n) \int_M \left(L_{(q)} + (2q-1)! \langle \left[T_{(2q-1)\alpha}, A_\beta \right] \cdot e^\top_\alpha, e^\top_\beta \rangle \right) \frac{1}{\sqrt{G}} dM \\ & + \frac{1}{2}(2q-1)! c_q(n) \int_\Sigma \left(\frac{|Df|^2}{1+|Df|^2} \right)^q H_{(2q-1)} d\Sigma. \end{aligned} \tag{33}$$

This finishes the proof of Theorem 2.

Let us now prove Theorem 3. We cannot use equation (33) directly, since, by hypothesis,

$$|Df| \to \infty \quad \text{as} \quad x \to \Sigma,$$

and hence, it is not possible to extend f to a smooth function on some open set containing $\mathbb{R}^n \setminus \Omega$. To circumvent this problem, we proceed as in the last section of [24]. Namely, we consider an approximating sequence

$$F^k = \left(f^{1,k}, \ldots, f^{m,k} \right) : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^m,$$

$k \in \mathbb{N}$, of smooth maps such that each F^k extends to a smooth map on some open set containing $\mathbb{R}^n \setminus \Omega$. We then apply (33) to each F^k and take the limit as $k \to \infty$, reaching (16).

It remains to prove inequality (17). If Σ has only one component then, by a result of Guan and Li [15], it holds

$$\frac{1}{2}(2q-1)! c_q(n) \int_\Sigma H_{(2q-1)} d\Sigma \geq \frac{1}{2^q} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2q}{n-1}}, \tag{34}$$

with equality holding if and only if Σ is a round sphere.

Suppose now that Σ has more than one component. Recall that if $x_1, \ldots, x_m$ are nonnegative real numbers and $0 \leq s < 1$ then

$$\sum_{i=1}^m x_i^s \geq \left(\sum_{i=1}^m x_i \right)^s, \tag{35}$$

with equality holding if and only if at most one of the x_i's is positive (see [20], Proposition 5.2). Inequality (17) then follows by combining inequalities (34) and (35).

Remark 5. Unfortunately our methods are not suitable to deal with the equality cases, that is, to prove the rigidity statements contained in conjectures 3 and 4. If equality holds in Theorem 1, then we can only conclude that the Gauss–Bonnet curvature $L_{(q)}$ is identically zero. If equality holds in Theorem 3, then we can only conclude that $L_{(q)}$ is zero on M and that Σ has only one component which is a round sphere.

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